1617-257 Fri Sep 23, Hour 6: Topology in R^n, Compactness.

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HW1

Riddle Along: Can you find uncountably many subsets of $\mathbb{N}$, s.t. the intersection of any two of them is finite?

Potatos: Hint: make them out of ghostium

Read Along: Munkres sec. 3, 4

Def $f: X \to Y$ is cont. at $x_0 \in X$ if for every nbd V of $f(x_0)$ [:= an open set containing $f(x_0)$] there is a nbd U of x_0 s.t. $f(U) \subset V$

$\iff \forall \epsilon > 0 \; \exists \delta > 0 \; d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon$

Def $f: X \to Y$ is cont. means $\forall x_0 \in X$, f is cont. at x_0.

Thm TFAE for $f: X \to Y$:

1. f is continuous.
2. For every V open in Y, $f^{-1}(V)$ is open in X
3. For every F closed in Y, $f^{-1}(F)$ is closed in X
4. if $X = Y = \mathbb{R}$, f is cont. in the 157 sense.

pre-write

Thm 1. constant functions are continuous.

2. $I: X \to X$ is cont.

3. $f: X \to Y$ cont, $A \subset X \Rightarrow f|_A : A \to Y$ is cont.

4. $f: X \to Y$, $g: Y \to Z \Rightarrow f /\!/ g = g \circ f$ is cont.

5. $f: X \to \mathbb{R}^n$ is $(f_1(x), f_2(x) \dots f_n(x))$.
Then f is cont $\iff \forall i \; f_i$ is cont.

6. $f, g: X \to \mathbb{R}$ cont $\Rightarrow f+g, f \cdot g, f-g, \frac{f}{g}$ (where defined) cont.

7. $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is cont.

Example: $f: \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^x$ is cont.

Skipped (but required): $f: X \to Y$, $\lim_{x \to x_0} f(x)$

.... Some theorems ....

<u>Def</u> $\text{int } A =$ union of all open sets contained in A $= \{x \in A : \exists \epsilon > 0 \; U(x,\epsilon) \subset A\}$

$\text{Ext } A = \text{int } A^c =$ union of all open sets disjoint from A. $\quad \text{Bd } A = X \setminus (\text{int } A \cup \text{ext } A)$

<u>Claim</u> $\text{int } A = \overline{(X \setminus A)}$, $\text{Ext } A = X \setminus \overline{A}$ $\quad \text{Bd } A = \overline{A} \cap \overline{(X \setminus A)}$

<u>Def</u> A set/space X is called "compact", if whenever you cover X with open sets, finitely many of these already cover X:

$$\left(X \subset \bigcup_{\alpha \in I} U_\alpha, \; U_\alpha \text{ open} \right) \Longrightarrow \left(\exists F \subset I \text{ finite, s.t. } X \subset \bigcup_{\alpha \in F} U_\alpha \right)$$

<u>Examples</u> 1. finite is compact.
2. $\mathbb{R}$ is not compact.
3. $(0,1)$ is not compact.

<u>Thm</u> $I = [0,1]$ is compact.

<u>Pf</u> Suppose U_α, $\alpha \in I$ is an open cover of I.
Let $B = \{y \in I : [0,y]$ can be covered by finitely many of the U_α's$\}$

Then $0 \in B$ and B is bounded. Let $b = \sup B$.
1. $b \in B$
2. $b = 1$ $\qquad \square$